Applied Mathematics I

Diploma Engineering E-content

Contents

Preface

Applied Mathematics forms the backbone of engineering education. The objective of this E-content is to develop analytical ability, logical thinking, and problem-solving skills required in engineering.

The contents include Algebra, Trigonometry, Complex Numbers and Coordinate Geometry with numerous illustrative examples and exercises.

Chapter 1

Algebra

1.1 Introduction

Algebra is one of the fundamental branches of mathematics. It provides techniques for solving equations, simplifying expressions, and modelling engineering problems.

The major topics covered in this chapter are

- Permutations and Combinations
- Binomial Theorem
- Partial Fractions
- Determinants
- Matrices

1.2 Permutations

A permutation is an arrangement of objects in a definite order.

If n distinct objects are available and r objects are selected, then the number of possible permutations is

$$^{n}P_r = \frac{n!}{(n-r)!},$$

where

$$n! = n(n-1)(n-2)\cdots 2\cdot 1.$$

1.3 Example

Find

$$^{8}P_3.$$

Solution
Using the formula,

$$^{8}P_3 = \frac{8!}{5!} = 8\times 7\times 6 = 336.$$

Hence,

$$\boxed{{}^{8}P_3 = 336.}$$

1.4 Combinations

A combination is a selection of objects in which the order of selection is not important.

If n distinct objects are available and r objects are selected, then the number of possible combinations is

$$^{n}C_r = \frac{n!}{r!(n-r)!},$$

where

$$0 \leq r \leq n.$$

1.4.1 Properties of Combinations

The following identities are frequently used.

$$^{n}C_0 = 1,$$

$$^{n}C_n = 1,$$

$$^{n}C_1 = n,$$

$$^{n}C_r = {}^{n}C_{n-r},$$

$$^{n}P_r = {}^{n}C_r\, r!.$$

1.5 Examples

Example 1

Find

$$^{10}C_3.$$

Solution

Using the formula,

$$^{10}C_3 = \frac{10!}{3!\,7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

Hence,

$$\boxed{{}^{10}C_3 = 120.}$$

Example 2

Evaluate

$$^{9}P_4.$$

Solution

$$^9P_4 = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024.$$

Therefore,

$$\boxed{^9P_4 = 3024.}$$

Example 3
Show that

$$^nP_r = {}^nC_r\, r!.$$

Solution
Since

$$^nC_r = \frac{n!}{r!(n-r)!},$$

multiplying both sides by $r!$,

$$^nC_r\, r! = \frac{n!}{(n-r)!} = {}^nP_r.$$

Hence proved.

1.6 Exercises

Evaluate the following.

1.
$$^7P_2$$

2.
$$^8C_4$$

3.
$$^{12}P_3$$

4.
$$^{15}C_2$$

5. Prove that
$$^nC_r = {}^nC_{n-r}.$$

6. If
$$^nC_2 = 28,$$
find the value of n.

1.7 Binomial Theorem

The Binomial Theorem provides the expansion of expressions of the form

$$(a+b)^n,$$

where n is a positive integer.
The expansion is

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r.$$

That is,

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n.$$

1.7.1 General Term

The $(r+1)$-th term of the expansion is

$$\boxed{T_{r+1} = {}^nC_r a^{n-r} b^r.}$$

1.8 Binomial Theorem

The Binomial Theorem provides the expansion of expressions of the form

$$(a+b)^n,$$

where n is a positive integer.
The theorem states that

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r.$$

Expanding,

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n.$$

The coefficients

$${}^nC_0,\ {}^nC_1,\ {}^nC_2,\ \ldots,\ {}^nC_n$$

are called the **binomial coefficients**.

1.8.1 General Term

The $(r+1)$-th term in the expansion of

$$(a+b)^n$$

is

$$\boxed{T_{r+1} = {}^nC_r a^{n-r} b^r.}$$

This formula helps in finding any required term directly without expanding the entire expression.

1.8.2 Middle Term

If n is even, then the expansion contains

$$n+1$$

terms and there is one middle term.
If n is odd, then there are two middle terms.

1.9 Examples

Example 1
Expand

$$(a+b)^4.$$

Solution
Using the Binomial Theorem,

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4.$$

Since

$${}^4C_0 = 1, \qquad {}^4C_1 = 4, \qquad {}^4C_2 = 6, \qquad {}^4C_3 = 4, \qquad {}^4C_4 = 1,$$

we obtain

$$\boxed{(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.}$$

Example 2
Expand

$$(x+2)^5.$$

Solution
Applying the Binomial Theorem,

$$(x+2)^5 = x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5.$$

Therefore,

$$\boxed{(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32.}$$

Example 3
Find the fourth term in the expansion of

$$(2x-3)^8.$$

Solution
The fourth term is obtained by putting

$$r = 3$$

in the general term.
Hence,

$$T_4 = {}^8C_3(2x)^5(-3)^3.$$

Since

$$^{8}C_3 = 56,$$

we have

$$T_4 = 56 \times 32x^5 \times (-27).$$

Thus,

$$\boxed{T_4 = -48384x^5.}$$

Example 4
Find the coefficient of

$$x^3$$

in the expansion of

$$(1+x)^7.$$

Solution
The coefficient of

$$x^3$$

is

$$^{7}C_3 = 35.$$

Hence,

$$\boxed{35.}$$

1.10 Exercises

Expand the following.

1. $$(a+b)^5$$

2. $$(x+y)^6$$

3. $$(2x+1)^4$$

4. $$(x-3)^5$$

5. $$(1+x)^8$$

Find the indicated terms.

6. Fourth term in

$$(3x+2)^7.$$

7. Coefficient of

$$x^4$$

in

$$(1+x)^9.$$

8. Middle term in

$$(a+b)^8.$$

9. General term in

$$(x+y)^{12}.$$

1.11 Binomial Expansion for Any Index

For any real number n satisfying

$$|x| < 1,$$

the expansion of

$$(1+x)^n$$

is

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots}$$

This expansion is called the generalized binomial expansion and is valid whenever

$$|x| < 1.$$

1.12 Partial Fractions

A rational function is the quotient of two polynomials.

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials and

$$Q(x) \neq 0.$$

If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational function is called a **proper rational function**. Otherwise, it is called an **improper rational function**. An improper fraction should first be converted into a proper fraction by polynomial division.

The process of expressing a proper rational function as the sum of simpler rational functions is called **partial fraction decomposition**.

1.12.1 Case I: Distinct Linear Factors

If

$$Q(x) = (x-a)(x-b)(x-c),$$

where all the factors are distinct, then

$$\boxed{\frac{P(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}.}$$

The constants A, B, and C are determined by comparing coefficients or by substituting suitable values of x.

1.12.2 Example 1

Resolve

$$\frac{5x+1}{(x-1)(x+2)}$$

into partial fractions.
Solution
Assume

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Multiplying both sides by

$$(x-1)(x+2),$$

we obtain

$$5x+1 = A(x+2) + B(x-1).$$

Putting

$$x = 1,$$

gives

$$6 = 3A,$$

therefore

$$A = 2.$$

Putting

$$x = -2,$$

gives

$$-9 = -3B,$$

therefore

$$B = 3.$$

Hence,

$$\boxed{\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}.}$$

1.12.3 Case II: Repeated Linear Factors

If the denominator contains repeated linear factors,

$$(x-a)^n,$$

then

$$\boxed{\frac{P(x)}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.}$$

1.12.4 Example 2

Resolve

$$\frac{2x+5}{(x+1)^2}$$

into partial fractions.
Solution
Assume

$$\frac{2x+5}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}.$$

Multiplying throughout by

$$(x+1)^2,$$

gives

$$2x+5 = A(x+1) + B.$$

Putting

$$x = -1,$$

we obtain

$$B = 3.$$

Comparing coefficients of x,

$$A = 2.$$

Therefore,

$$\boxed{\frac{2x+5}{(x+1)^2} = \frac{2}{x+1} + \frac{3}{(x+1)^2}.}$$

1.12.5 Case III: Irreducible Quadratic Factors

If the denominator contains a quadratic factor which cannot be factorised,

$$x^2 + ax + b,$$

then

$$\boxed{\frac{P(x)}{x^2 + ax + b} = \frac{Ax + B}{x^2 + ax + b}.}$$

Similarly, if the quadratic factor is repeated,

$$(x^2 + ax + b)^2,$$

then

$$\boxed{\frac{P(x)}{(x^2 + ax + b)^2} = \frac{A_1x + B_1}{x^2 + ax + b} + \frac{A_2x + B_2}{(x^2 + ax + b)^2}.}$$

1.13 Exercises

Resolve the following into partial fractions.

1. $$\frac{7x + 5}{(x + 1)(x + 3)}$$

2. $$\frac{x + 4}{(x - 2)(x + 5)}$$

3. $$\frac{3x + 8}{(x + 2)^2}$$

4. $$\frac{2x^2 + 5x + 3}{(x + 1)(x + 2)(x + 3)}$$

5. $$\frac{x^2 + 1}{(x - 1)^2(x + 2)}$$

1.14 Determinants

A determinant is a square array of numbers enclosed within vertical lines. Determinants play an important role in solving systems of linear equations, finding inverses of matrices, and many engineering applications.

A determinant of order two is written as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

and its value is

$$\boxed{ad - bc.}$$

1.14.1 Determinant of Order Two

A determinant of order two is represented as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Its value is obtained by taking the difference of the products of the principal and secondary diagonals.

Thus,

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.}$$

1.14.2 Example 1

Evaluate

$$\begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix}.$$

Solution
Using the formula,

$$\begin{vmatrix} 3 & 5 \\ 2 & 7 \end{vmatrix} = 3(7) - 5(2) = 21 - 10 = 11.$$

Hence,

$$\boxed{11}$$

1.14.3 Determinant of Order Three

A determinant of order three is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The value of a third-order determinant may be obtained by Laplace expansion or by Sarrus' rule.

1.15 Laplace Expansion

The expansion of a determinant along the first row is

$$\boxed{|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},}$$

where

$$A_{ij} = (-1)^{i+j}M_{ij},$$

and

$$M_{ij}$$

denotes the minor of the element

$$a_{ij}.$$

1.16 Minors

The minor of an element is the determinant obtained by deleting the row and the column containing that element.

For example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

the minor corresponding to

$$a_{11}$$

is

$$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48 = -3.$$

1.17 Cofactors

The cofactor of an element is defined by

$$\boxed{A_{ij} = (-1)^{i+j} M_{ij}.}$$

Thus,

$$A_{11} = M_{11},$$

$$A_{12} = -M_{12},$$

$$A_{13} = M_{13},$$

and so on.

1.17.1 Example 2

Evaluate

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix}$$

using Laplace expansion.

Solution

Expanding along the first row,

$$= 1 \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2 \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}.$$

Now,

$$= 1(0 - 24) - 2(0 - 20) + 3(0 - 5).$$

Hence,

$$= -24 + 40 - 15 = 1.$$

Therefore,

$$\boxed{\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{vmatrix} = 1.}$$

1.18 Properties of Determinants

The following properties are frequently used.

1. Interchanging any two rows (or columns) changes the sign of the determinant.
2. If any two rows (or columns) are identical, then the determinant is zero.
3. If every element of a row (or column) is multiplied by a constant k, then the determinant is also multiplied by k.
4. If every element of a row (or column) is zero, then the determinant is zero.
5. The value of a determinant remains unchanged if a multiple of one row is added to another row.

1.19 Exercises

Evaluate the following determinants.

1.
$$\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$$

2.
$$\begin{vmatrix} 6 & 2 \\ 1 & 7 \end{vmatrix}$$

3.
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix}$$

4.
$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

5. Find the minors and cofactors of all elements of
$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}.$$

1.20 Matrices

A matrix is a rectangular arrangement of numbers, symbols or algebraic expressions arranged in rows and columns.

A matrix having m rows and n columns is called an

$$m \times n$$

matrix and is represented by

$$A = [a_{ij}]_{m\times n},$$

where

$$a_{ij}$$

denotes the element lying in the i-th row and j-th column.

1.20.1 Types of Matrices

Matrices are classified according to their order and the arrangement of their elements.

Row Matrix

A matrix having only one row is called a row matrix.

For example,

$$A = \begin{bmatrix} 2 & 5 & 7 & 9 \end{bmatrix}.$$

Column Matrix

A matrix having only one column is called a column matrix.

For example,

$$B = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 10 \end{bmatrix}.$$

Rectangular Matrix

A matrix having different numbers of rows and columns is called a rectangular matrix.

For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Square Matrix

A matrix having the same number of rows and columns is called a square matrix.

For example,

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

Diagonal Matrix

A square matrix in which all non-diagonal elements are zero is called a diagonal matrix.
For example,

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Scalar Matrix

A diagonal matrix in which all diagonal elements are equal is called a scalar matrix.
For example,

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Identity Matrix

A square matrix whose principal diagonal elements are unity and all other elements are zero is called an identity matrix.
It is denoted by

$$I.$$

For example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Zero Matrix

A matrix in which every element is zero is called a zero matrix.
For example,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

1.21 Equality of Matrices

Two matrices are said to be equal if

1. They have the same order.
2. Their corresponding elements are equal.

Thus,

$$A = [a_{ij}]$$

and

$$B = [b_{ij}]$$

are equal if

$$a_{ij} = b_{ij}$$

for every

$$i \quad \text{and} \quad j.$$

1.22 Addition of Matrices

Two matrices can be added only when they have the same order.

If

$$A = [a_{ij}]$$

and

$$B = [b_{ij}],$$

then

$$\boxed{A + B = [a_{ij} + b_{ij}].}$$

1.22.1 Example 1

Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix}.$$

Find

$$A + B.$$

Solution

Adding the corresponding elements,

$$A + B = \begin{bmatrix} 2+1 & 3+6 \\ 4+2 & 5+4 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 6 & 9 \end{bmatrix}.$$

Hence,

$$\boxed{A + B = \begin{bmatrix} 3 & 9 \\ 6 & 9 \end{bmatrix}.}$$

1.23 Subtraction of Matrices

If

$$A = [a_{ij}]$$

and

$$B = [b_{ij}]$$

are matrices of the same order, then

$$\boxed{A - B = [a_{ij} - b_{ij}].}$$

1.23.1 Example 2

Find

$$A - B,$$

where

$$A = \begin{bmatrix} 7 & 5 \\ 6 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Solution
Subtracting corresponding elements,

$$A - B = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}.$$

Hence,

$$\boxed{A - B = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}.}$$

1.24 Multiplication of Matrices

Two matrices can be multiplied only when the number of columns of the first matrix is equal to the number of rows of the second matrix.

Suppose

$$A = [a_{ij}]_{m\times n}$$

and

$$B = [b_{ij}]_{n\times p},$$

then the product

$$AB$$

is defined and is a matrix of order

$$m \times p.$$

If

$$C = AB = [c_{ij}],$$

then each element of C is obtained by

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.}$$

1.24.1 Example

Multiply

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Solution
The element in the first row and first column is

$$1 \times 5 + 2 \times 7 = 19.$$

The element in the first row and second column is

$$1 \times 6 + 2 \times 8 = 22.$$

The element in the second row and first column is

$$3 \times 5 + 4 \times 7 = 43.$$

The element in the second row and second column is

$$3 \times 6 + 4 \times 8 = 50.$$

Hence,

$$\boxed{AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.}$$

1.25 Properties of Matrix Multiplication

If the products are defined, then the following properties hold.

1. $$A(BC) = (AB)C.$$

2. $$A(B + C) = AB + AC.$$

3. $$(A + B)C = AC + BC.$$

4. Matrix multiplication is generally not commutative, that is,

$$AB \neq BA.$$

5. $$AI = IA = A,$$

   where I is the identity matrix.

1.26 Transpose of a Matrix

The transpose of a matrix is obtained by interchanging its rows and columns.

If

$$A = [a_{ij}],$$

then the transpose of A is denoted by

$$A^T$$

and is defined by

$$\boxed{A^T = [a_{ji}].}$$

1.26.1 Example

Find the transpose of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solution

Interchanging rows and columns,

$$\boxed{A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.}$$

1.27 Adjoint of a Matrix

The transpose of the cofactor matrix is called the **adjoint** (or adjugate) of the matrix.

It is denoted by

$$\text{adj}(A).$$

Thus,

$$\boxed{\text{adj}(A) = (A_{ij})^T,}$$

where A_{ij} represents the cofactors of the elements of A.

1.28 Inverse of a Matrix

A square matrix A is said to be invertible if there exists another matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

The inverse of a matrix is given by

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}, \qquad |A| \neq 0.}$$

Hence, a matrix possesses an inverse only if its determinant is non-zero.

1.28.1 Example

Find the inverse of

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

Solution
The determinant is

$$|A| = 2(3) - 1(5) = 1.$$

The adjoint matrix is

$$\operatorname{adj}(A) = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.$$

Therefore,

$$\boxed{A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}.}$$

1.29 Solution of Linear Equations by Cramer's Rule

Consider the system of equations

$$a_1x + b_1y = c_1,$$

$$a_2x + b_2y = c_2.$$

Let

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

If

$$\Delta \neq 0,$$

then the solution is

$$\boxed{x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta},}$$

where

$$\Delta_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix},$$

and

$$\Delta_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

1.29.1 Example

Solve

$$2x + y = 7,$$

$$3x + 2y = 12.$$

Solution
Here,

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1.$$

Also,

$$\Delta_x = \begin{vmatrix} 7 & 1 \\ 12 & 2 \end{vmatrix} = 14 - 12 = 2,$$

and

$$\Delta_y = \begin{vmatrix} 2 & 7 \\ 3 & 12 \end{vmatrix} = 24 - 21 = 3.$$

Hence,

$$\boxed{x = 2, \qquad y = 3.}$$

1.30 Exercises

1. Multiply

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}.$$

2. Find the transpose of

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}.$$

3. Find the inverse of

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

4. Solve by Cramer's rule

$$3x + 2y = 8,$$

$$5x - y = 7.$$

Chapter 2

Trigonometry

2.1 Introduction

Trigonometry is the branch of mathematics that deals with the relationships between the sides and angles of triangles. It has extensive applications in engineering, surveying, navigation, architecture, physics and astronomy.

The topics covered in this chapter include

- Trigonometric Ratios
- Allied Angles
- Compound Angles
- Multiple and Sub-multiple Angles
- Product Formulae

2.2 Trigonometric Ratios

Trigonometry is based on the six trigonometric ratios of an acute angle.

Consider a right-angled triangle ABC, right-angled at B.

Let

- Hypotenuse = AC
- Perpendicular = BC
- Base = AB

For an angle θ,

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}.$$

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}.$$

$$\tan\theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{\sin\theta}{\cos\theta}.$$

The remaining three ratios are

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{Base}}{\text{Perpendicular}},$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{Hypotenuse}}{\text{Base}},$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{Hypotenuse}}{\text{Perpendicular}}.$$

2.3 Standard Trigonometric Ratios

The values of trigonometric ratios for standard angles are given below.

Angle	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not Defined
$\cot\theta$	Not Defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not Defined
$\csc\theta$	Not Defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

2.4 Fundamental Identities

The following identities are frequently used in trigonometry.

$$\boxed{\sin^2\theta + \cos^2\theta = 1.}$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta.}$$

$$\boxed{1 + \cot^2\theta = \csc^2\theta.}$$

2.5 Allied Angles

Angles related to

$$0^\circ,\ 90^\circ,\ 180^\circ,\ 270^\circ$$

are called allied angles.
Some useful formulae are

$$\sin(90^\circ - \theta) = \cos\theta,$$

$$\cos(90^\circ - \theta) = \sin\theta,$$

$$\tan(90^\circ - \theta) = \cot\theta,$$

$$\cot(90^\circ - \theta) = \tan\theta,$$

$$\sec(90^\circ - \theta) = \csc\theta,$$

$$\csc(90^\circ - \theta) = \sec\theta.$$

2.6 Signs of Trigonometric Ratios

The sign of a trigonometric ratio depends upon the quadrant.

Quadrant	Positive Ratios
First	All ratios
Second	sin, csc
Third	tan, cot
Fourth	cos, sec

This is remembered by the rule

$$\boxed{\text{ASTC}}$$

which stands for

All, Sine, Tangent, Cosine.

2.7 Examples

Example 1
Evaluate

$$\sin 30^\circ + \cos 60^\circ.$$

Solution
Since

$$\sin 30^\circ = \frac{1}{2},$$

and

$$\cos 60^\circ = \frac{1}{2},$$

we obtain

$$\boxed{\sin 30^\circ + \cos 60^\circ = 1.}$$

Example 2
Evaluate

$$\tan 45^\circ + \cot 45^\circ.$$

Solution
Since

$$\tan 45^\circ = 1,$$

and

$$\cot 45^\circ = 1,$$

therefore

$$\boxed{\tan 45^\circ + \cot 45^\circ = 2.}$$

Example 3
Prove that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Solution
Using the definitions,

$$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}},$$

and

$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}.$$

Hence,

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{Perpendicular}}{\text{Hypotenuse}}}{\frac{\text{Base}}{\text{Hypotenuse}}} = \frac{\text{Perpendicular}}{\text{Base}} = \tan\theta.$$

Therefore,

$$\boxed{\frac{\sin\theta}{\cos\theta} = \tan\theta.}$$

2.8 Exercises

1. Evaluate

$$\sin 45^\circ + \cos 45^\circ.$$

2. Evaluate

$$\tan 60^\circ - \cot 30^\circ.$$

3. Prove that

$$\sec^2\theta - \tan^2\theta = 1.$$

4. Prove that

$$\csc^2\theta - \cot^2\theta = 1.$$

5. Find

$$\sin(90^\circ - 35^\circ).$$

2.9 Compound Angle Formulae

The trigonometric ratios of the sum and difference of two angles can be expressed in terms of the trigonometric ratios of the individual angles.

2.9.1 Sine of Sum and Difference

The angle sum formula is

$$\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B.}$$

The angle difference formula is

$$\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B.}$$

2.9.2 Cosine of Sum and Difference

The angle sum formula is

$$\boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B.}$$

The angle difference formula is

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B.}$$

2.9.3 Tangent of Sum and Difference

The tangent of the sum of two angles is

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}.}$$

Similarly,

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.}$$

2.10 Examples

Example 1
Evaluate

$$\sin 75^\circ.$$

Solution
Since

$$75^\circ = 45^\circ + 30^\circ,$$

we obtain

$$\sin 75^\circ = \sin(45^\circ + 30^\circ).$$

Using the angle sum formula,

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ.$$

Substituting the standard values,

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}.$$

Hence,

$$\boxed{\sin 75^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.}$$

Example 2
Evaluate

$$\cos 15^\circ.$$

Solution
Since

$$15^\circ = 45^\circ - 30^\circ,$$

we obtain

$$\cos 15^\circ = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ.$$

Therefore,

$$\boxed{\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}.}$$

2.11 Product Formulae

The following product identities are useful in simplifying trigonometric expressions.

$$\boxed{2 \sin A \cos B = \sin(A + B) + \sin(A - B).}$$

$$\boxed{2 \cos A \cos B = \cos(A + B) + \cos(A - B).}$$

$$\boxed{2 \sin A \sin B = \cos(A - B) - \cos(A + B).}$$

2.12 Transformation Formulae

The above identities can also be written in reverse form.

$$\boxed{\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right).}$$

$$\boxed{\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right).}$$

$$\boxed{\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right).}$$

$$\boxed{\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right).}$$

2.13 Double Angle Formulae

Putting

$$A = B = \theta,$$

in the compound angle formulae, we obtain

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta.}$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta.}$$

Alternative forms are

$$\boxed{\cos 2\theta = 2\cos^2\theta - 1.}$$

and

$$\boxed{\cos 2\theta = 1 - 2\sin^2\theta.}$$

Also,

$$\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.}$$

2.14 Triple Angle Formulae

The triple angle identities are

$$\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta.}$$

$$\boxed{\cos 3\theta = 4\cos^3\theta - 3\cos\theta.}$$

$$\boxed{\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.}$$

2.15 Half Angle Formulae

The half angle identities are

$$\boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}.}$$

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}.}$$

$$\boxed{\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}.}$$

2.16 Exercises

1. Find
$$\sin 75^\circ.$$

2. Find
$$\cos 15^\circ.$$

3. Find
$$\tan 75^\circ.$$

4. Prove that
$$\sin 2A = 2\sin A\cos A.$$

5. Prove that
$$\cos 2A = 1 - 2\sin^2 A.$$

6. Find
$$\sin 60^\circ \cos 30^\circ.$$

7. Express
$$2\sin A\cos B$$
as the sum of two trigonometric functions.

8. Find
$$\sin 3\theta$$
in terms of
$$\sin\theta.$$

9. Find

$$\cos 3\theta$$

in terms of

$$\cos \theta.$$

10. Find

$$\tan 2\theta$$

in terms of

$$\tan \theta.$$

Chapter 3

Complex Numbers

3.1 Introduction

The concept of complex numbers was introduced to obtain the solutions of equations such as

$$x^2 + 1 = 0,$$

which has no real solution.
A complex number is of the form

$$\boxed{z = a + ib,}$$

where

- a is called the **real part**,
- b is called the **imaginary part**,
- $i = \sqrt{-1}$ is called the imaginary unit.

Thus,

$$\mathrm{Re}(z) = a,$$

and

$$\mathrm{Im}(z) = b.$$

3.2 Imaginary Unit

The imaginary unit is denoted by

$$i = \sqrt{-1},$$

and satisfies

$$i^2 = -1.$$

Higher powers of i are

$$i^3 = -i,$$

$$i^4 = 1,$$

$$i^5 = i,$$

$$i^6 = -1,$$

and the pattern repeats after every four powers.

3.3 Equality of Complex Numbers

Two complex numbers

$$a + ib$$

and

$$c + id$$

are equal if

$$a = c$$

and

$$b = d.$$

3.4 Addition of Complex Numbers

If

$$z_1 = a + ib,$$

and

$$z_2 = c + id,$$

then

$$\boxed{z_1 + z_2 = (a + c) + i(b + d).}$$

3.4.1 Example

Add

$$(3 + 5i)$$

and

$$(2 - 4i).$$

Solution

$$(3 + 5i) + (2 - 4i) = 5 + i.$$

Hence,

$$\boxed{5 + i.}$$

3.5 Subtraction of Complex Numbers

If

$$z_1 = a + ib,$$

and

$$z_2 = c + id,$$

then

$$\boxed{z_1 - z_2 = (a - c) + i(b - d).}$$

3.5.1 Example

Subtract

$$(2 + 3i)$$

from

$$(7 + 8i).$$

Solution

$$(7 + 8i) - (2 + 3i) = 5 + 5i.$$

Hence,

$$\boxed{5 + 5i.}$$

3.6 Multiplication of Complex Numbers

If

$$z_1 = a + ib,$$

and

$$z_2 = c + id,$$

then

$$\boxed{z_1 z_2 = (ac - bd) + i(ad + bc).}$$

3.6.1 Example

Multiply

$$(2 + 3i)$$

and

$$(4 - i).$$

Solution

$$(2 + 3i)(4 - i) = 8 - 2i + 12i - 3i^2.$$

Since

$$i^2 = -1,$$

we obtain

$$= 8 + 10i + 3.$$

Hence,

$$\boxed{11 + 10i.}$$

3.7 Division of Complex Numbers

To divide two complex numbers, multiply the numerator and denominator by the conjugate of the denominator.

If

$$z_1 = a + ib,$$

and

$$z_2 = c + id,$$

then

$$\boxed{\frac{z_1}{z_2} = \frac{(a + ib)(c - id)}{c^2 + d^2}.}$$

3.7.1 Example

Find

$$\frac{3 + 4i}{1 - 2i}.$$

Solution

Multiply the numerator and denominator by

$$1 + 2i.$$

Thus,

$$\frac{3+4i}{1-2i} = \frac{(3+4i)(1+2i)}{1+4}.$$

Now,

$$(3+4i)(1+2i) = 3+6i+4i+8i^2.$$

Since

$$i^2 = -1,$$

we obtain

$$= -5+10i.$$

Therefore,

$$\boxed{\frac{3+4i}{1-2i} = -1+2i.}$$

3.8 Conjugate of a Complex Number

If

$$z = a+ib,$$

then the complex number

$$\boxed{\overline{z} = a-ib}$$

is called the **conjugate** of z.
The conjugate of a complex number is obtained by changing the sign of its imaginary part.

3.8.1 Properties of Conjugates

For any two complex numbers z_1 and z_2,

$$\boxed{\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}.}$$

$$\boxed{\overline{z_1-z_2} = \overline{z_1}-\overline{z_2}.}$$

$$\boxed{\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}.}$$

$$\boxed{\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, \qquad z_2 \neq 0.}$$

3.8.2 Example

Find the conjugate of

$$5-7i.$$

Solution
Changing the sign of the imaginary part,

$$\boxed{\overline{z} = 5+7i.}$$

3.9 Modulus of a Complex Number

The modulus (or absolute value) of a complex number

$$z = a + ib$$

is denoted by

$$|z|$$

and is defined as

$$\boxed{|z| = \sqrt{a^2 + b^2}.}$$

The modulus is always a non-negative real number.

3.9.1 Example

Find the modulus of

$$3 + 4i.$$

Solution
Using the definition,

$$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$

Hence,

$$\boxed{|3 + 4i| = 5.}$$

3.10 Argument of a Complex Number

The angle made by the line joining the origin to the point representing the complex number with the positive real axis is called the **argument** of the complex number.
It is denoted by

$$\arg(z) = \theta.$$

If

$$z = a + ib,$$

then

$$\boxed{\tan\theta = \frac{b}{a}.}$$

The value of the argument depends upon the quadrant in which the point lies.

3.10.1 Example

Find the argument of

$$1 + i.$$

Solution
Here,

$$a = 1, \qquad b = 1.$$

Hence,

$$\tan\theta = 1.$$

Therefore,

$$\boxed{\theta = 45^\circ = \frac{\pi}{4}.}$$

3.11 Polar Form of a Complex Number

If

$$|z| = r$$

and

$$\arg(z) = \theta,$$

then the complex number may be written as

$$\boxed{z = r(\cos\theta + i\sin\theta).}$$

This representation is known as the **polar form** (or trigonometric form) of a complex number.

3.12 Conversion from Cartesian Form to Polar Form

Suppose

$$z = a + ib.$$

The conversion is carried out using the following steps.

1. Find

$$r = \sqrt{a^2 + b^2}.$$

2. Find

$$\theta = \tan^{-1}\left(\frac{b}{a}\right),$$

taking the correct quadrant into consideration.

3. Write

$$z = r(\cos\theta + i\sin\theta).$$

3.12.1 Example

Express

$$1 + i$$

in polar form.
Solution
The modulus is

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

The argument is

$$\theta = 45^\circ.$$

Hence,

$$\boxed{1 + i = \sqrt{2}\left(\cos 45^\circ + i\sin 45^\circ\right).}$$

3.13 Conversion from Polar Form to Cartesian Form

If

$$z = r(\cos\theta + i\sin\theta),$$

then

$$\boxed{z = r\cos\theta + ir\sin\theta.}$$

Thus,

$$a = r\cos\theta, \qquad b = r\sin\theta.$$

3.14 Exercises

1. Find the conjugate of

$$6 - 5i.$$

2. Find the modulus of

$$4 - 3i.$$

3. Find the argument of

$$\sqrt{3} + i.$$

4. Express

$$2 + 2i$$

in polar form.

5. Convert

$$5\left(\cos 30^\circ + i \sin 30^\circ\right)$$

into Cartesian form.

6. Find

$$(3 + 2i) + (4 - 5i).$$

7. Find

$$(5 + 3i)(2 - i).$$

8. Evaluate

$$\frac{6 + 8i}{2 - i}.$$

Chapter 4

Coordinate Geometry

4.1 Introduction

Coordinate Geometry is the branch of mathematics in which geometrical figures are represented analytically with the help of coordinates.

It establishes a relationship between algebra and geometry and has extensive applications in engineering drawing, mechanics, computer graphics, surveying, and navigation.

The topics covered in this chapter are

- Straight Line
- Circle
- Parabola
- Ellipse
- Hyperbola

4.2 Distance Between Two Points

Let

$$P(x_1, y_1)$$

and

$$Q(x_2, y_2)$$

be two points in the Cartesian plane.

The distance between the two points is given by

$$\boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

4.2.1 Example

Find the distance between

$$(2, 3)$$

and

$$(6, 8).$$

Solution
Using the distance formula,

$$PQ = \sqrt{(6-2)^2 + (8-3)^2} = \sqrt{16+25} = \sqrt{41}.$$

Hence,

$$\boxed{PQ = \sqrt{41}.}$$

4.3 Section Formula

Let

$$P(x_1, y_1)$$

and

$$Q(x_2, y_2)$$

be two points.
If a point

$$R(x, y)$$

divides the line joining P and Q internally in the ratio

$$m : n,$$

then

$$\boxed{x = \frac{mx_2 + nx_1}{m+n}, \qquad y = \frac{my_2 + ny_1}{m+n}.}$$

4.3.1 Midpoint Formula

If

$$m = n,$$

then R is the midpoint of the line segment.
Hence,

$$\boxed{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).}$$

4.3.2 Example

Find the midpoint of

$$(4, 7)$$

and

$$(8, 3).$$

Solution
Using the midpoint formula,

$$x = \frac{4+8}{2} = 6,$$

$$y = \frac{7+3}{2} = 5.$$

Hence,

$$\boxed{(6, 5).}$$

4.4 Slope of a Straight Line

The slope of a straight line is defined as the ratio of the vertical change to the horizontal change.
If the line passes through

$$(x_1, y_1)$$

and

$$(x_2, y_2),$$

then

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}.}$$

4.4.1 Example

Find the slope of the line joining

$$(2, 5)$$

and

$$(8, 11).$$

Solution
Using the formula,

$$m = \frac{11-5}{8-2} = \frac{6}{6} = 1.$$

Hence,

$$\boxed{m = 1.}$$

4.5 Equation of a Straight Line

A straight line can be represented in different standard forms.

4.5.1 Slope-Intercept Form

If

$$m$$

is the slope and

$$c$$

is the intercept on the y-axis, then

$$\boxed{y = mx + c.}$$

4.5.2 Point-Slope Form

A line passing through

$$(x_1, y_1)$$

with slope

$$m$$

is

$$\boxed{y - y_1 = m(x - x_1).}$$

4.5.3 Two-Point Form

A line passing through

$$(x_1, y_1)$$

and

$$(x_2, y_2)$$

is

$$\boxed{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.}$$

4.5.4 Intercept Form

If a line cuts intercepts

$$a$$

and

$$b$$

on the coordinate axes, then

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1.}$$

4.5.5 General Form

The most general equation of a straight line is

$$\boxed{Ax + By + C = 0,}$$

where

$$A$$

and

$$B$$

are not both zero.

4.5.6 Example

Find the equation of the line passing through

$$(2, 3)$$

having slope

$$4.$$

Solution
Using the point-slope form,

$$y - 3 = 4(x - 2).$$

Therefore,

$$y = 4x - 5.$$

Hence,

$$\boxed{4x - y - 5 = 0.}$$

4.6 Angle Between Two Straight Lines

If the slopes of two lines are

$$m_1$$

and

$$m_2,$$

then the angle between them is

$$\boxed{\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}.}$$

If

$$m_1 m_2 = -1,$$

the two lines are perpendicular.
If

$$m_1 = m_2,$$

the two lines are parallel.

4.7 Exercises

1. Find the distance between
$$(3, 4)$$
and
$$(7, 9).$$

2. Find the midpoint of
$$(5, 6)$$
and
$$(9, 12).$$

3. Find the slope of the line joining
$$(1, 2)$$
and
$$(5, 10).$$

4. Find the equation of the line passing through
$$(4, -2)$$
having slope
$$3.$$

5. Find the angle between the lines whose slopes are
$$2$$
and
$$-1.$$

4.8 Intersection of Two Straight Lines

Consider the two straight lines

$$a_1x + b_1y + c_1 = 0,$$

and

$$a_2x + b_2y + c_2 = 0.$$

The point satisfying both equations simultaneously is called the **point of intersection** of the two lines.

The coordinates of the point of intersection may be obtained by solving the two equations simultaneously.

4.8.1 Example

Find the point of intersection of

$$2x + y - 7 = 0$$

and

$$x - y - 2 = 0.$$

Solution
From

$$x - y = 2,$$

we obtain

$$y = x - 2.$$

Substituting into the first equation,

$$2x + x - 2 = 7,$$

which gives

$$3x = 9,$$

therefore

$$x = 3.$$

Hence,

$$y = 1.$$

Thus,

$$\boxed{(3, 1)}$$

is the point of intersection.

4.9 Perpendicular Distance of a Point from a Line

Let the equation of a straight line be

$$Ax + By + C = 0,$$

and let

$$(x_1, y_1)$$

be a point outside the line.
The perpendicular distance of the point from the line is

$$\boxed{d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.}$$

4.9.1 Example

Find the perpendicular distance of the point

$$(2, 3)$$

from the line

$$3x + 4y - 10 = 0.$$

Solution
Using the formula,

$$d = \frac{|3(2) + 4(3) - 10|}{\sqrt{3^2 + 4^2}} = \frac{8}{5}.$$

Hence,

$$\boxed{d = \frac{8}{5}.}$$

Chapter 5

Circle

5.1 Introduction

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point remains constant.

The fixed point is called the **centre** and the constant distance is called the **radius**.

5.2 Standard Equation of a Circle

If the centre of a circle is

$$(h, k)$$

and the radius is

$$r,$$

then its equation is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2.}$$

5.2.1 Example

Find the equation of the circle having centre

$$(2, -1)$$

and radius

$$5.$$

Solution
Using the standard equation,

$$(x-2)^2 + (y+1)^2 = 25.$$

Hence,

$$\boxed{(x-2)^2 + (y+1)^2 = 25.}$$

5.3 General Equation of a Circle

Expanding the standard equation, we obtain

$$\boxed{x^2 + y^2 + 2gx + 2fy + c = 0,}$$

where

$$g,\ f,\ c$$

are constants.
The centre of the circle is

$$\boxed{(-g, -f),}$$

and the radius is

$$\boxed{r = \sqrt{g^2 + f^2 - c}.}$$

5.3.1 Example

Find the centre and radius of

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

Solution
Comparing with

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we obtain

$$g = -3, \qquad f = 2, \qquad c = -12.$$

Hence,

$$\boxed{\text{Centre} = (3, -2).}$$

Also,

$$r = \sqrt{(-3)^2 + 2^2 - (-12)} = \sqrt{25} = 5.$$

Therefore,

$$\boxed{\text{Radius} = 5.}$$

5.4 Equation of a Circle Passing Through the End Points of a Diameter

Let the end points of a diameter be

$$(x_1, y_1)$$

and

$$(x_2, y_2).$$

The equation of the circle is

$$\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.}$$

5.4.1 Example

Find the equation of the circle having

$$(2, 3)$$

and

$$(6, 7)$$

as the end points of a diameter.
Solution
Using the formula,

$$(x - 2)(x - 6) + (y - 3)(y - 7) = 0.$$

Expanding,

$$x^2 + y^2 - 8x - 10y + 33 = 0.$$

Hence,

$$\boxed{x^2 + y^2 - 8x - 10y + 33 = 0.}$$

5.5 Equation of a Circle Passing Through Three Points

Let the three points be

$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3).$$

Assume the equation of the circle as

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Substituting the coordinates of the three given points gives three simultaneous equations in

$$g,\ f,\ c,$$

which may be solved to determine the required circle.

5.6 Exercises

1. Find the equation of the circle having centre

   $$(3, -2)$$

   and radius

   $$4.$$

2. Find the centre and radius of

   $$x^2 + y^2 + 8x - 6y - 11 = 0.$$

3. Find the equation of the circle whose diameter has end points

$$(1, 2)$$

and

$$(5, 6).$$

4. Find the perpendicular distance of the point

$$(3, 5)$$

from the line

$$2x - y + 7 = 0.$$

Chapter 6

Conic Sections

6.1 Introduction

A conic section is the curve obtained by the intersection of a plane with a right circular cone.

The four important conic sections are

- Circle
- Parabola
- Ellipse
- Hyperbola

6.2 Parabola

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point is always equal to its perpendicular distance from a fixed straight line.

The fixed point is called the **focus** and the fixed straight line is called the **directrix**.

6.2.1 Standard Equation

If the vertex of the parabola is at the origin and the axis is along the positive x-axis, then its equation is

$$\boxed{y^2 = 4ax,}$$

where a is the distance of the focus from the vertex.

The focus is

$$(a, 0),$$

and the directrix is

$$x = -a.$$

6.2.2 Example

Find the focus and directrix of

$$y^2 = 16x.$$

Solution
Comparing with

$$y^2 = 4ax,$$

we obtain

$$4a = 16.$$

Hence,

$$a = 4.$$

Therefore,

$$\boxed{\text{Focus} = (4, 0),}$$

and

$$\boxed{\text{Directrix} : x = -4.}$$

6.3 Ellipse

An ellipse is the locus of a point such that the sum of its distances from two fixed points remains constant.

The fixed points are called the foci of the ellipse.

6.3.1 Standard Equation

The standard equation of an ellipse having centre at the origin is

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b.}$$

The coordinates of the foci are

$$(\pm c, 0),$$

where

$$c^2 = a^2 - b^2.$$

6.3.2 Example

Find the lengths of the semi-major axis and semi-minor axis of

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Solution
Comparing with the standard equation,

$$a^2 = 25, \qquad b^2 = 9.$$

Hence,

$$\boxed{a = 5, \qquad b = 3.}$$

6.4 Hyperbola

A hyperbola is the locus of a point such that the difference of its distances from two fixed points remains constant.

6.4.1 Standard Equation

The standard equation of a hyperbola having centre at the origin is

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

The foci are

$$(\pm c, 0),$$

where

$$c^2 = a^2 + b^2.$$

6.4.2 Example

Identify the conic represented by

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

Solution
Since the equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the curve represents a

$$\boxed{\text{Hyperbola.}}$$

6.5 Comparison of Conic Sections

Conic	Standard Equation	Main Property
Circle	$x^2 + y^2 = r^2$	Constant radius
Parabola	$y^2 = 4ax$	Focus and directrix
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Sum of distances is constant
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Difference of distances is constant

6.6 Solved Problems

Problem 1
Find the focus of

$$y^2 = 20x.$$

Solution

Comparing with

$$y^2 = 4ax,$$

we obtain

$$4a = 20.$$

Hence,

$$a = 5.$$

Therefore,

$$\boxed{\text{Focus} = (5, 0).}$$

Problem 2
Find the centre and radius of

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

Solution
Comparing with

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

we obtain

$$g = -2, \qquad f = 3, \qquad c = -12.$$

Hence,

$$\boxed{\text{Centre} = (2, -3).}$$

Also,

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 + 12} = 5.$$

Therefore,

$$\boxed{\text{Radius} = 5.}$$

6.7 Exercises

1. Find the focus and directrix of

$$y^2 = 24x.$$

2. Find the centre and radius of

$$x^2 + y^2 - 8x + 4y - 5 = 0.$$

3. Find the equation of the circle having centre

$$(2, 5)$$

and radius

$$6.$$

4. Identify the following conics:

(a)

$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

(b)

$$\frac{x^2}{49} - \frac{y^2}{25} = 1$$

(c)

$$y^2 = 12x$$

5. Find the equation of the straight line passing through

$$(4, -1)$$

having slope

$$2.$$

Summary

In this E-content we have covered the following topics:

- Algebra
- Permutations and Combinations
- Binomial Theorem
- Partial Fractions
- Determinants
- Matrices
- Trigonometry
- Complex Numbers
- Coordinate Geometry
- Straight Line
- Circle
- Conic Sections

Recommended Books

1. B. S. Grewal, *Elementary Engineering Mathematics*, Khanna Publishers.
2. H. K. Dass, *Engineering Mathematics*, S. Chand Publications.
3. R. D. Sharma, *Applied Mathematics*, Dhanpat Rai Publications.
4. S. Kohli, *Engineering Mathematics*, IPH Publications.
5. S. S. Sabharwal and Sunita Jain, *Applied Mathematics*, Eagle Prakashan.